

# SCREENING IN HIGH- $T$ QCD

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These days, as high energy particle colliders become unavailable for testing speculative theoretical ideas, physicists are looking to other environments that may provide extreme conditions where theory confronts physical reality. One such circumstance may arise at high temperature  $T$ , which perhaps can be attained in heavy ion collisions or in astrophysical settings. It is natural therefore to examine the high-temperature behavior of the standard model, and here I shall report on recent progress in constructing the high- $T$  limit of QCD.

My presentation will be unified by the theme of **screening**, a familiar phenomenon in electrodynamical plasmas. I shall explore how similar effects can be described in QCD at a sufficiently high temperature (above the putative confinement - deconfinement phase transition) so that we may speak of unconfirmed quarks and gluons forming a plasma. But first let me review briefly the screening phenomena in plasmas of electromagnetically charged particles. We begin with Poisson's equation, which relates the scalar electric potential  $\phi$  to a charge density  $\rho$ .

$$-\nabla^2 \phi = \rho$$

For the charge density we take a statistical distribution of positive-charged ( $+q$ ) and negative-charged ( $-q$ ) particles, each carrying the energy  $\pm q\phi$ , respectively, and described by the same density  $n$ . Then

$$\rho = n(qe^{-q\phi/T} - qe^{q\phi/T})$$

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For large  $T$ , this becomes

$$\rho \underset{(\text{large } T)}{\sim} -2nq^2\phi/T$$

so that the Poisson equation reads

$$-\nabla^2\phi + \left(\frac{2nq^2}{T}\right)\phi = O$$

Evidently, a screening mass  $\propto \left(\frac{nq^2}{T}\right)^{\frac{1}{2}}$  has been induced for the electric potential  $\phi$ ; the inverse is called the **Debye screening length**. Again at high  $T$  and for a relativistic plasma, one expects  $n \sim (1/\text{volume}) \sim T^3$ , hence the induced **electric** screening mass is

$$m \propto |q|T$$

We shall see a similar result emerging in the non-Abelian theory as well. Note that Debye screening occurs for the electric (temporal) component of the gauge potential. There is no electro-dynamical magnetic screening, because there are no magnetic resources.

$$\nabla \cdot \mathbf{B} = O$$

In the non-Abelian theory, the corresponding equation involves the covariant divergence.

$$\nabla \cdot \mathbf{B}^a = gf^{abc}\mathbf{A}^b \cdot \mathbf{B}^c$$

(Here  $g$  is the gauge coupling constant.) So the issue of magnetic sources is not so clear in the Yang-Mills case, and one of the topics that we shall address later is whether in the non-Abelian theory there exists magnetic screening.

The above argument – it is essentially Debye’s – makes little use of field theoretical formalism. But to carry through analogous calculations in the standard model, we shall begin with quantum field theory. Let me explore how finite-temperature calculations are performed in that context.

When studying a field theory at finite temperature, the simplest approach is the so-called imaginary-time formalism. We continue time to the imaginary interval  $[0, 1/iT]$  and consider bosonic (fermionic) fields to be periodic (anti-periodic) on that interval. Perturbative calculations are performed by the usual Feynman rules as at zero temperature, except that in the conjugate energy-momentum, Fourier-transformed space, the energy variable  $p^0$  (conjugate to the periodic time variable) becomes discrete – it is  $2\pi nT$ , ( $n$  integer) for bosons. From this one immediately sees that at high temperature – in the limiting case, at infinite temperature – the time direction disappears, because the temporal interval shrinks to zero. Only zero-energy processes survive, since “non-vanishing energy” necessarily means high energy owing to the discreteness of the energy variable  $p^0 \sim 2\pi nT$ , and therefore all modes with  $n \neq 0$  decouple at large  $T$ . In this way a Euclidean three-dimensional field theory becomes effective at high temperatures and describes essentially static processes.<sup>1</sup>

Let me repeat in greater detail. Finite- $T$ , imaginary-time perturbation theory makes use of conventional diagrammatic analysis in “momentum” space, with modified “energy” variables, as indicated above. Specifically a spinless boson propagator is

$$D(p) = \frac{i}{p_0^2 - \mathbf{p}^2 - m^2} \qquad p_0 = i\pi(2n)T$$

while a spin- $\frac{1}{2}$  fermion propagator reads

$$S(p) = \frac{i}{\gamma^0 p_0 - \boldsymbol{\gamma} \cdot \mathbf{p} - m} \quad p_0 = i\pi(2n+1)T$$

The zero-temperature integration measure  $\int \frac{d^4 p}{(2\pi)^4}$  becomes replaced by  $iT \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3}$ . Thus it is seen that Bose exchange between two  $O(g)$  vertices contributes  $iT \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} g \frac{i}{-4\pi^2 n^2 T^2 - \mathbf{p}^2 - m^2} g$  where  $g$  is the coupling strength. In the large  $T$  limit, all  $n \neq 0$  terms (formally) vanish as  $T^{-1}$  and only the  $n = 0$  term survives. One is left with  $\int \frac{d^3 p}{(2\pi)^3} g \sqrt{T} \frac{1}{\mathbf{p}^2 + m^2} g \sqrt{T}$ . This is a Bose exchange graph in a Euclidian 3-dimensional theory, with effective coupling  $g\sqrt{T}$ . Similar reasoning leads to the conclusion that fermions decouple at large  $T$ .

While all this is quick and simple, it may be physically inadequate. First of all, frequently one is interested in non-static processes in real time, so complicated analytic continuation from imaginary time needs to be made before passing to the high- $T$  limit, which in imaginary time describes only static processes. Also one may wish to study amplitudes where the real external energy is neither large nor zero, even though virtual internal energies are high.

Another reason that the above may be inadequate emerges when we consider massless fields (such as those that occur in QCD). We have seen that the  $n = 0$  mode leaves a propagator that behaves as  $\frac{1}{\mathbf{p}^2}$  when mass vanishes, and a phase space of  $d^3 p$ . It is well known that this kind of kinematics at low momenta leads to infrared divergences in perturbation theory even for off-mass-shell amplitudes — Green's functions in massless Bosonic field theories possess infrared divergences in naive perturbation theory.<sup>2</sup> Since physical QCD does not suffer from off-mass-shell infrared divergences, perturbation theory must be resummed.

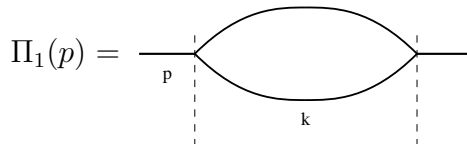
A final shortcoming of the above limiting procedure is that it is formal: the limit is taken before the integration/summation is carried out. But the latter need not converge uniformly; indeed owing to ultraviolet divergences, it may not converge at all and must be renormalized. As a consequence the  $n \neq 0$  contributions in single Boson exchange graphs may not decrease as  $T^{-1}$ .

Thus the formal arguments for the emergence of a 3-dimensional theory at high- $T$  need be re-examined for QCD. Nevertheless, even if unreliable, the arguments alert us to the possibility that 3-dimensional field theoretic structures may emerge in the high- $T$  regime. Indeed this occurs, although not in a direct, straightforward fashion; this will be demonstrated presently.

Here is a graphical argument to the same end discussed above: *viz.* The need to resum perturbation theory. Consider a one-loop amplitude  $\Pi_1(p)$ ,

$$\Pi_1(p) \equiv \int dk I_1(p, k) ,$$

given by the graph in the figure.



$$\equiv \int dk I_1(p, k)$$

Compare this to a two-loop amplitude  $\Pi_2(p)$ ,

$$\Pi_2(p) \equiv \int dk I_2(p, k) ,$$

in which  $\Pi_1$  is an insertion, as in the figure below.

$$\begin{aligned} \Pi_2(p) &= \text{Diagram: A horizontal line with a vertex on the left labeled 'p' and a vertex on the right. A loop is attached to the left vertex, consisting of a circle with a vertical dashed line passing through its center. The loop is labeled 'g' at the top and 'k' at the bottom. A vertical dashed line also passes through the right vertex.} \\ &\equiv \int dk I_2(p, k) \end{aligned}$$

Following Pisarski,<sup>3</sup> we estimate the relative importance of  $\Pi_2$  to  $\Pi_1$  by the ratio of their integrands,

$$\frac{\Pi_2}{\Pi_1} \sim \frac{I_2}{I_1} = g^2 \frac{\Pi_1(k)}{k^2} ,$$

Here  $g$  is the coupling constant, and the  $k^2$  in the denominator reflects the fact that we are considering a massless field, as in QCD. Clearly the  $k^2 \rightarrow 0$  limit is relevant to the question whether the higher order graph can be neglected relative to the lower order one. Because one finds that for small  $k$  and large  $T$ ,  $\Pi_1(k)$  behaves as  $T^2$ , the ratio  $\Pi_2/\Pi_1$  is  $g^2 T^2/k^2$ . As a result when  $k$  is  $O(gT)$  or smaller the two-loop amplitude is not negligible compared to the one-loop amplitude. Thus graphs with “soft” external momenta [ $O(gT)$  or smaller] have to be included as insertions in higher order calculations.

A terminology has arisen: graphs with generic/soft external moment [ $O(gT)$  where  $g$  is small and  $T$  is large] and large internal momenta [the internal momenta are integration variables in an amplitude; when  $T$  is large they are  $O(T)$ , hence also large] are called “hard thermal loops.”<sup>3,4</sup> Much study has been expended on them and finally a general picture has emerged. Before presenting general results, let us look at a specific example — a 2-point Green’s function.

It needs to be appreciated that in the imaginary-time formalism the correlation functions are unique and definite. But passage to real time, requires continuing from the integer-valued “energy” to a continuous variable, and this cannot be performed uniquely. This reflects the fact that in real time there exists a variety of correlation functions: time ordered products, retarded commutators, advanced commutators, *etc.* Essentially one is seeing the consequence of the fact that a Euclidean Laplacian possesses a unique inverse, whereas giving an inverse for the Minkowskian d’Alembertian requires specifying temporal boundary conditions, and a variety of answers can be gotten with a variety of boundary conditions. Thus, when presenting results one needs to specify precisely what one is computing.

We shall consider a correlation function for two fermionic currents, in the 1-loop approximation.

$$\Pi^{\mu\nu}(x, y) = -i \langle j^\mu(x) j^\nu(y) \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \Pi^{\mu\nu}(k)$$

The QCD result differs from the QED result by a group theoretical multiplicative factor, so we present high- $T$  results only for the latter, in real-time, and consider the time-ordered product  $\Pi_T^{\mu\nu}$  as well as the retarded commutator  $\Pi_R^{\mu\nu}$ .

$\Pi^{\mu\nu}$  possesses a real and an imaginary part. It is found that at large  $T$ , the real parts of  $\Pi_T^{\mu\nu}$  and  $\Pi_R^{\mu\nu}$  coincide.

$$-\text{Re}\Pi^{\mu\nu}(k) = \frac{T^2}{6} P_2^{\mu\nu} + \frac{T^2 k^2}{|\mathbf{k}^2|} \left[ 1 + \frac{k^0}{2|\mathbf{k}|} \ln \left| \frac{k^0 - |\mathbf{k}|}{k^0 + |\mathbf{k}|} \right| \right] \left[ \frac{1}{3} P_1^{\mu\nu} + \frac{1}{2} P_2^{\mu\nu} \right]$$

where the projection operators are

$$P_1^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2$$

$$P_2^{\mu\nu} = \begin{cases} 0 & \text{if } \mu \text{ or } \nu = 0 \\ \delta^{ij} - k^i k^j / |\mathbf{k}|^2 & \text{otherwise} \end{cases}$$

For the imaginary part, which is present only for space-like arguments, different expressions are found.

$$-\text{Im}\Pi_R^{\mu\nu}(k) \equiv -\pi \rho^{\mu\nu}(k)$$

$$= \frac{\pi k^2}{|\mathbf{k}|^3} \frac{k^0}{2} T^2 \theta(-k^2) \left[ \frac{1}{3} P_1^{\mu\nu} + \frac{1}{2} P_2^{\mu\nu} \right]$$

$$-\text{Im}\Pi_T^{\mu\nu}(k) = \frac{\pi k^2}{|\mathbf{k}|^3} \left[ \frac{k^0}{2} T^2 + T^3 \right] \theta(-k^2) \left[ \frac{1}{3} P_1^{\mu\nu} + \frac{1}{2} P_2^{\mu\nu} \right]$$

A unified presentation of these formulas is achieved in a dispersive representation. For the retarded function this reads

$$\Pi_R^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - k_0 - i\epsilon}$$

while the time-ordered expression is

$$\Pi_T^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - k_0 - i\epsilon} + \frac{2\pi i}{e^{k_0/T} - 1} \rho^{\mu\nu}(k_0, \mathbf{k})$$

The dispersive expressions may also be used to give the imaginary-time formula.

$$\Pi_{\text{imaginary time}}^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - 2\pi i n T}$$

In all the above formulas,  $\Pi_{SUB}^{\mu\nu}$  is a real subtraction term.

Note that a universal statement about high- $T$  behavior can be made only for the absorptive part  $\rho^{\mu\nu}$ : it is  $O(T^2)$ . This also characterizes  $\Pi_R^{\mu\nu}$ , but  $\Pi_T^{\mu\nu}$  possesses an additional  $O(T^3)$  imaginary part, which is seen to arise from the additional term in  $\Pi_T^{\mu\nu}$  involving the bosonic distribution function  $\frac{1}{e^{k_0/T} - 1}$ . Finally, the  $\Pi_{\text{imaginary time}}^{\mu\nu}$  amplitude has a temperature

behavior determined by its external “energy”  $= 2\pi i n T$ . If this is replaced by a fixed  $k_0$  ( $T$ -independent) or if only the  $n = 0$  mode is considered, then one may assign an  $O(T^2)$  behavior to this quantity as well.

In conclusion, we assert that the 2-point correlation function behaves as  $O(T^2)$ , where it is understood that this statement is to be applied to the retarded amplitude, or to the imaginary time amplitude with its “energy” argument continued away from  $2\pi i n T$ .<sup>5</sup>

Similar analysis has been performed on the higher-point functions and this work has culminated with the discovery (Braaten, Pisarski, Frenkel, Taylor)<sup>3,4,6</sup> of a remarkable simplicity in their structure. To describe this simplicity, we do not discuss the individual  $n$ -point functions, but rather their sum multiplied by powers of the vector potential, *viz.* we consider the generating functional for single-particle irreducible Green’s functions with gauge field external lines in the hard thermal limit. (Effectively, we are dealing with continued imaginary-time amplitudes.) We call this quantity  $\Gamma_{\text{HTL}}(A)$  and it is computed in an  $SU(N)$  gauge theory containing  $N_F$  fermion species of the fundamental representation.  $\Gamma_{\text{HTL}}$  is found (i) to be proportional to  $(N + \frac{1}{2}N_F)$ , (ii) to behave as  $T^2$  at high temperature, and (iii) to be gauge invariant.

$$\begin{aligned}
 & \text{A wavy line with a circle} + \text{A wavy line with a circle and a wavy line} + \text{A wavy line with a circle and two wavy lines} + \text{A wavy line with a circle and three wavy lines} + \dots \\
 & = (N + \frac{1}{2}N_F) \frac{g^2 T^2}{12\pi} \Gamma_{\text{HTL}}(A) \\
 & \Gamma_{\text{HTL}}(U^{-1} A U + U^{-1} dU) = \Gamma_{\text{HTL}}(A)
 \end{aligned}$$

(Henceforth  $g$ , the coupling constant, is scaled to unity.) A further kinematical simplification in  $\Gamma_{\text{HTL}}$  has also been established. To explain this we define two light-like four-vectors  $Q_{\pm}^{\mu}$  depending on a unit three-vector  $\hat{q}$ , pointing in an arbitrary direction.

$$Q_{\pm}^{\mu} = \frac{1}{\sqrt{2}}(1, \pm \hat{q})$$

$$\hat{q} \cdot \hat{q} = 1, \quad Q_{\pm}^{\mu} Q_{\pm\mu} = 0, \quad Q_{\pm}^{\mu} Q_{\mp\mu} = 1$$

Coordinates and potentials are projected onto  $Q_{\pm}^{\mu}$ .

$$x^{\pm} \equiv x_{\mu} Q_{\pm}^{\mu}, \quad \partial_{\pm} \equiv Q_{\pm}^{\mu} \frac{\partial}{\partial x^{\mu}}, \quad A_{\pm} \equiv A_{\mu} Q_{\pm}^{\mu}$$

The additional fact that is now known is that (iv) after separating an ultralocal contribution from  $\Gamma_{\text{HTL}}$ , the remainder may be written as an average over the angles of  $\hat{q}$  of a functional  $W$  that depends only on  $A_{+}$ ; also this functional is non-local only on the two-dimensional  $x^{\pm}$  plane, and is ultralocal in the remaining directions, perpendicular to the  $x^{\pm}$  plane. [“Ultralocal” means that any potentially non-local kernel  $k(x, y)$  is in fact a  $\delta$ -function of the difference  $k(x, y) \propto \delta(x - y)$ .]

$$\Gamma_{\text{HTL}}(A) = 2\pi \int d^4x A_0^a(x) A_0^a(x) + \int d\Omega_{\hat{q}} W(A_{+})$$

These results are established in perturbation theory, and a perturbative expansion of  $W(A_+)$ , *i.e.* a power series in  $A_+$ , exhibits the above mentioned properties. A natural question is whether one can sum the series to obtain an expression for  $W(A_+)$ .

Important progress on this problem was made when it was observed (Taylor, Wong)<sup>6</sup> that the gauge-invariance condition can be imposed infinitesimally, whereupon it leads to a functional differential equation for  $W(A_+)$ , which is best presented as

$$\begin{aligned} \frac{\partial}{\partial x^+} \frac{\delta}{\delta A_+^a} \left[ W(A_+) + \frac{1}{2} \int d^4x A_+^b(x) A_+^b(x) \right] \\ - \frac{\partial}{\partial x^-} \left[ A_+^a \right] + f^{abc} A_+^b \frac{\delta}{\delta A_+^c} \left[ W(A_+) + \frac{1}{2} \int d^4x A_+^d(x) A_+^d(x) \right] = 0 \end{aligned}$$

In other words we seek a quantity, call it

$$S(A_+) \equiv W(A_+) + \frac{1}{2} \int d^4x A_+^a(x) A_+^a(x) ,$$

which is a functional on a two-dimensional manifold  $\{x^+, x^-\}$ , depends on a single functional variable  $A_+$ , and satisfies

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f^{abc} A_1^b \frac{\delta}{\delta A_1^c} S = 0$$

$$“1” \equiv x^+ , \quad “2” \equiv -x^- , \quad A_1^a \equiv A_+^a$$

Another suggestive version of the above is gotten by defining  $A_2^a \equiv \frac{\delta S}{\delta A_1^a}$ .

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0$$

To solve the functional equation and produce an expression for  $W(A_+)$ , we now turn to a completely different corner of physics, and that is Chern-Simons theory at zero temperature.

The Chern-Simons term is a peculiar gauge theoretic topological structure that can be constructed in odd dimensions, and here we consider it in 3-dimensional space-time.

$$I_{CS} \propto \int d^3x \epsilon^{\alpha\beta\gamma} \text{Tr} (\partial_\alpha A_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma)$$

This object was introduced into physics over a decade ago, and since that time it has been put to various physical and mathematical uses. Indeed one of our originally stated motivations for studying the Chern-Simons term was its possible relevance to high-temperature gauge theory.<sup>7</sup> Here following Efraty and Nair,<sup>8</sup> we shall employ the Chern-Simons term for a determination of the hard thermal loop generating functional,  $\Gamma_{\text{HTL}}$ .

Since it is the space-time integral of a density,  $I_{CS}$  may be viewed as the action for a quantum field theory in (2+1)-dimensional space-time, and the corresponding Lagrangian would then be given by a two-dimensional, spatial integral of a Lagrange density.

$$\begin{aligned} I_{CS} &\propto \int dt L_{CS} \\ L_{CS} &\propto \int d^2x \left( A_2^a \dot{A}_1^a + A_0^a F_{12}^a \right) \end{aligned}$$

I have separated the temporal index (0) from the two spatial ones (1,2) and have indicated time differentiation by an over dot.  $F_{12}^a$  is the non-Abelian field strength, defined on a two-dimensional plane.

$$F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c$$

Examining the Lagrangian, we see that it has the form

$$L \sim p\dot{q} - \lambda H(p, q)$$

where  $A_2^a$  plays the role of  $p$ ,  $A_1^a$  that of  $q$ ,  $F_{12}^a$  is like a Hamiltonian and  $A_0^a$  acts like the Lagrange multiplier  $\lambda$ , which forces the Hamiltonian to vanish; here  $A_0^a$  enforces the vanishing of  $F_{12}^a$ .

$$F_{12}^a = 0$$

The analogy instructs us how the Chern-Simons theory should be quantized.

We postulate equal-time computation relations, like those between  $p$  and  $q$ .

$$[A_1^a(\mathbf{r}), A_2^b(\mathbf{r}')] = i \delta^{ab} \delta(\mathbf{r} - \mathbf{r}')$$

In order to satisfy the condition enforced by the Lagrange multiplier, we demand that  $F_{12}^a$ , operating on “allowed” states, annihilate them.

$$F_{12}^a | \rangle = 0$$

This equation can be explicitly presented in a Schrödinger-like representation for the Chern-Simons quantum field theory, where the state is a functional of  $A_1^a$ . The action of the operators  $A_1^a$  and  $A_2^a$  is by multiplication and functional differentiation, respectively.

$$\begin{aligned} | \rangle &\sim \Psi(A_1^a) \\ A_1^a | \rangle &\sim A_1^a \Psi(A_1^a) \\ A_2^a | \rangle &\sim \frac{1}{i} \frac{\delta}{\delta A_1^a} \Psi(A_1^a) \end{aligned}$$

This, of course, is just the field theoretic analog of the quantum mechanical situation where states are functions of  $q$ , the  $q$  operator acts by multiplication, and the  $p$  operator by differentiation. In the Schrödinger representation, the condition that states be annihilated by  $F_{12}^a$

$$\left( \partial_1 A_2^a - \partial_2 A_1^a + f_{abc} A_1^b A_2^c \right) | \rangle = 0$$

leads to a functional differential equation.

$$\left( \partial_1 \frac{1}{i} \frac{\delta}{\delta A_1^a} - \partial_2 A_1^a + f_{abc} A_1^b \frac{1}{i} \frac{\delta}{\delta A_1^c} \right) \Psi(A_1^a) = 0$$

If we define  $S$  by  $\Psi = e^{iS}$  we get equivalently

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f_{abc} A_1^b \frac{\delta}{\delta A_1^c} S = 0$$



This equation comprises the entire content of Chern-Simons quantum field theory.  $S$  is the Chern-Simons eikonal, which gives the exact wave functional owing to the simple dynamics of the theory. Also the above eikonal equation is recognized to be precisely the equation for the hard thermal loop generating functional, given above.

Let me elaborate on the connection with eikonal-WKB ideas. Let us recall that in particle quantum mechanics, when the wave function  $\psi(q)$  is written in eikonal form

$$\psi(q) = e^{iS(q)}$$

then the WKB approximation to  $S(q)$  is given by the integral of the canonical 1-form  $p dq$

$$S(q) = \int^q p(q') dq'$$

where  $p(q)$ , the momentum, is taken to be function of the coordinate  $q$ , by virtue of satisfying the equation of motion.

$$\begin{aligned} \frac{p^2(q)}{2} + V(q) &= E \\ p(q) &= \sqrt{2E - 2V(q)} \end{aligned}$$

Analogously, in the present field theory application, the eikonal  $S(A_1)$  may be written as a functional integral,

$$S(A_1) = \int^{A_1} A_2^a(A'_1) \mathcal{D} A_1^a$$

where  $A_2^a(A_1)$  is functional of  $A_1$  determined by the equation of motion

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0$$

Since, by construction  $\frac{\delta S}{\delta A_1^a} = A_2^a$ , it is clear that as a consequence  $S$  satisfies the required equation. However, we reiterate that in the Chern-Simons case there is no WKB approximation: everything is exact owing to the simplicity of Chern-Simons dynamics.

The gained advantage for thermal physics is that “acceptable” Chern-Simons states, *i.e.* solutions to the above functional equations, were constructed long ago,<sup>9</sup> and one can now take over those results to the hard thermal loop problem. One knows from the Chern-Simons work that  $\Psi$  and  $S$  are given by a 2-dimensional fermionic determinant, *i.e.* by the Polyakov-Wiegman expression. While these are not described by very explicit formulas, many properties are understood, and the hope is that one can use these properties to obtain further information about high-temperature QCD processes. We give two applications.

The Chern-Simons information allows presenting the hard-thermal loop generating functional as

$$\Gamma_{\text{HTL}} = \frac{1}{2} \int d\Omega_{\hat{q}} [A_+^a A_-^a + S(A_+) + S(A_-)] \quad .$$

Using the known properties of  $S$ , one can give a very explicit series expansion for  $\Gamma_{\text{HTL}}$  in terms of powers of  $A$

$$\Gamma_{\text{HTL}} = \frac{1}{2!} \int \Gamma_{\text{HTL}}^{(2)} AA + \frac{1}{3!} \int \Gamma_{\text{HTL}}^{(3)} AAA + \dots$$

where the non-local kernels  $\Gamma_{\text{HTL}}^{(i)}$  are known explicitly. This power series may be used to systematize the resummation procedure for the perturbative theory. Here is what one does: perturbation theory for Green's functions may be organized with the help of a functional integral, where the integrand contains (among other factors)  $e^{iI_{\text{QCD}}(A)}$  where  $I_{\text{QCD}}$  is the QCD action. We now rewrite that as

$$e^{i\left\{I_{\text{QCD}}(A)+\frac{m^2}{4\pi}\Gamma_{\text{HTL}}(A)-\frac{m^2}{4\pi}\Gamma_{\text{HTL}}(A)\right\}}$$

where  $m = T\sqrt{\frac{N+N_F/2}{3}}$ . Obviously nothing has changed, because we have merely added and subtracted the hard-thermal-loop generating functional. Next we introduce a loop counting parameter  $l$ : in an  $l$ -expansion, different powers of  $l$  correspond to different numbers of loops, but at the end  $l$  is set to unity. The resummed action is then taken to be

$$e^{iI_{\text{resummed}}} = e^{i\left\{\frac{1}{l}\left[I_{\text{QCD}}(\sqrt{l}A)+\frac{m^2}{4\pi}\Gamma_{\text{HTL}}(\sqrt{l}A)\right]-\frac{m^2}{4\pi}\Gamma_{\text{HTL}}(\sqrt{l}A)\right\}}$$

One readily verifies that an expansion in powers of  $l$  describes the resummed perturbation theory, and this then represents the first application of the present Chern-Simons formalism.

For a second application, we note that even though the closed form for  $\Gamma_{\text{HTL}}$  is not very explicit, a much more explicit formula can be gotten for its functional derivative  $\frac{\delta\Gamma_{\text{HTL}}}{\delta A_\mu^a}$ . This may be identified with an induced current, which is then used as a source in the Yang-Mills equation. Thereby one obtains a non-Abelian generalization of the Kubo equation, which governs the response of the hot quark gluon plasma to external disturbances.<sup>5</sup>

$$D_\mu F^{\mu\nu} = \frac{m^2}{2} j_{\text{induced}}^\nu$$

From the known properties of the fermionic determinant — hard thermal loop generating functional — one can show that  $j_{\text{induced}}^\mu$  is given by

$$j_{\text{induced}}^\mu = \int \frac{d\Omega_{\hat{q}}}{4\pi} \left\{ Q_+^\mu \left( a_-(x) - A_-(x) \right) + Q_-^\mu \left( a_+(x) - A_+(x) \right) \right\}$$

where  $a_\pm$  are solutions to the equations

$$\begin{aligned} \partial_+ a_- - \partial_- A_+ + [A_+, a_-] &= 0 \\ \partial_+ A_- - \partial_- a_+ + [a_+, A_-] &= 0 \end{aligned}$$

Evidently  $j_{\text{induced}}^\mu$ , as determined by the above equations, is a non-local and non-linear functional of the vector potential  $A_\mu$ .

There now have appeared several alternative derivations of the Kubo equation. Blaizot and Iancu<sup>10</sup> have analyzed the Schwinger-Dyson equations in the hard thermal regime; they truncated them at the 1-loop level, made further kinematical approximations that are justified in the hard thermal limit, and they too arrived at the Kubo equation. Equivalently the argument may be presented succinctly in the language of the composite effective action,<sup>11</sup> which is truncated at the 1-loop (semi-classical) level — two-particle irreducible graphs are omitted. The stationarity condition on the 1-loop action is the gauge invariance constraint

on  $\Gamma_{\text{HTL}}$ . Finally, there is one more, entirely different derivation — which perhaps is the most interesting because it relies on classical physics.<sup>12</sup> We shall give the argument presently, but first we discuss solutions for the Kubo equation.

To solve the Kubo equation, one must determine  $a_{\pm}$  for arbitrary  $A_{\pm}$ , thereby obtaining an expression for the induced current, as a functional of  $A_{\pm}$ . Since the functional is non-local and non-linear, it does not appear possible to construct it explicitly in all generality. However, special cases can be readily handled.

In the Abelian case, everything commutes and linearizes. One can determine  $a_{\pm}$  in terms of  $A_{\pm}$ .

$$a_{\pm} = \frac{\partial_{\pm}}{\partial_{\mp}} A_{\mp}$$

(Incidentally, this formula exemplifies the kinematical simplicity, mentioned above, of hard thermal loops: the nonlocality of  $1/\partial_{\pm}$  lies entirely in the  $\{x^+, x^-\}$  plane.) With the above form for  $a_{\pm}$  inserted into the Kubo equation, the solution can be constructed explicitly. It coincides with the results obtained by Silin long ago, on the basis of the Boltzmann-Vlasov equation.<sup>13</sup> One sees that the present theory is the non-Abelian generalization of that physics; in particular  $m$ , given above, is recognized as the Debye screening length, which remains gauge invariant in the non-Abelian context.

It is especially interesting to emphasize that Silin did not use quantum field theory in his derivation; rather he employed classical transport theory. Nevertheless, his final result coincides with what here has been developed from a quantal framework. This raises the possibility that the non-Abelian Kubo equation can also be derived classically, and indeed such a derivation has been given, as mentioned above.

We now pause in our discussion of solutions to the non-Abelian Kubo equation in order to describe its classical derivation.

Transport theory is formulated in terms of a single-particle distribution function  $f$  on phase space. In the Abelian case,  $f$  depends on position  $\{x^{\mu}\}$  and momentum  $\{p^{\mu}\}$  of the particle. For the non-Abelian theory it is necessary to take into account the fact that the particle's non-Abelian charge  $\{Q^a\}$  also is a dynamical variable:  $Q^a$  satisfies an evolution equation (see below) and is an element of phase space. Therefore, the non-Abelian distribution function depends on  $\{x^{\mu}\}$ ,  $\{p^{\mu}\}$  and  $\{Q^a\}$ , and in the collisionless approximation obeys the transport equation  $\frac{d}{d\tau}f = 0$ , *i.e.*

$$\frac{\partial f}{\partial x^{\mu}} \frac{dx^{\mu}}{d\tau} + \frac{\partial f}{\partial p^{\mu}} \frac{dp^{\mu}}{d\tau} + \frac{\partial f}{\partial Q^a} \frac{dQ^a}{d\tau} = 0$$

The derivatives of the phase-space variables are given by the Wong equations, for a particle with mass  $\mu$ .

$$\begin{aligned} \frac{dx^{\mu}}{d\tau} &= \frac{p^{\mu}}{\mu} \\ \frac{dp^{\mu}}{d\tau} &= F_a^{\mu\nu} \frac{dx_{\nu}}{d\tau} Q^a \\ \frac{dQ^a}{d\tau} &= -f^{abc} \frac{dx^{\mu}}{d\tau} A_{\mu}^b Q^c \end{aligned}$$

In order to close the system we need an equation for  $F^{\mu\nu}$ . In a microscopic description (with a single particle) one would have  $(D_\mu F^{\mu\nu})^a = \int d\tau Q^a(\tau) \frac{p^\nu(\tau)}{\mu} \delta^4(x - x(\tau))$  and consistency would require covariant conservation of the current; this is ensured provided  $Q^a$  satisfies the equation given above. In our macroscopic, statistical derivation, the current is given in terms of the distribution function, so the system of equations closes with

$$(D_\mu F^{\mu\nu})^a = \int dp dQ Q^a p^\nu f(x, p, Q)$$

(One verifies that the current – the right side of the above – is covariantly conserved.) The collisionless transport equation, with the equations of motion inserted, is called the Boltzmann equation. The closed system formed by the latter supplemented with the Yang-Mills equation is known as the non-Abelian Vlasov equations. To make progress, this highly non-linear set of equations is approximated by expanding around the equilibrium form for  $f$ ,

$$\begin{array}{c} \text{free} \\ f_{\text{boson}} \\ \text{fermion} \end{array} \propto \left( e^{\frac{1}{T} \sqrt{\mathbf{p}^2 + \mu^2}} \mp 1 \right)^{-1}$$

This comprises the Vlasov approximation, and readily leads to the non-Abelian Kubo equation.<sup>12</sup>

One may say that the non-Abelian theory is the minimal elaboration of the Abelian case needed to preserve non-Abelian gauge invariance. The fact that classical reasoning can reproduce quantal results is presumably related to the fact that the quantum theory makes use of the (resummed) 1-loop approximation, which is frequently recognized as an essentially classical effect. Evidently, the quantum fluctuations included in the hard thermal loops coincide with thermal fluctuations.

Returning now to our summary of the solutions to the non-Abelian Kubo equation that have been obtained thus far, we mention first that the static problem may be solved completely.<sup>11</sup> When the *Ansatz* is made that the vector potential is time independent,  $A_\pm = A_\pm(\mathbf{r})$ , one may solve for  $a_\pm$  to find  $a_\pm = -A_\pm$  and the induced current is explicitly computed as

$$\frac{m^2}{2} j_{\text{induced}}^\mu = \begin{pmatrix} -m^2 A^0 \\ \mathbf{0} \end{pmatrix}$$

This exhibits gauge-invariant electric screening with Debye mass  $m$ . One may also search for localized static solutions to the Kubo equation, but one finds only infinite energy solutions, carrying a point-magnetic monopole singularity. Thus there are no plasma solitons in high-T QCD.<sup>11</sup> Specifically, upon selecting the radially symmetric solution that decreases at large distances, there arises a magnetic monopole-like singularity at the origin.

Much less is known concerning time-dependent solutions. Blaizot and Iancu<sup>14</sup> have made the *Ansatz* that the vector potentials depend only on the combination  $x \cdot k$ , where  $k$  is an arbitrary 4-vector:  $A_\pm = A_\pm(x \cdot k)$ . Once again  $a_\pm$  can be determined; one finds  $a_\pm = \frac{Q_\pm \cdot k}{Q_\mp \cdot k} A_\mp$ , and the induced current is computable. For  $k = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$ , where there is no space dependence (only a dependence on time is present) one finds

$$\frac{m^2}{2} j_{\text{induced}}^\mu = \begin{pmatrix} 0 \\ -\frac{1}{3} m^2 \mathbf{A} \end{pmatrix}$$

More complicated expressions hold with general  $k$ . The Kubo equation can be solved numerically; the resulting profile is a non-Abelian generalization of a plasma plane wave.

The physics of all these solutions, as well as of other, still undiscovered ones, remains to be elucidated, and I invite any of you to join in this interesting task.

We see that Debye electric screening is reproduced in essentially the same form as in an Abelian plasma (to leading order). How about magnetic screening? It is important to appreciate that the above time-independent, space-independent induced current, with  $\mathbf{j}$  proportional to  $\mathbf{A}$ , does **not** describe magnetic screening because screening is determined by static configurations. Thus we conclude that the hard-thermal-loop limit of hot QCD does not show magnetic screening. Indeed it appears that if one proceeds perturbatively, beyond the resummed perturbation expansion of hard thermal loops, no direct evidence for magnetic screening can be found.

However, there is indirect evidence: although the hard thermal loop resummation cures some of the perturbative infrared divergences, as one calculates to higher perturbative orders, they reappear essentially due to the non-linear interactions between electric (temporal) and unscreened magnetic (spatial) degrees of freedom as well as among the magnetic degrees of freedom due to their self-interaction. (Such interactions are absent in an Abelian theory.) Consequently it is believed that non-perturbative magnetic screening arises in the non-Abelian theory, and it is recalled that, as mentioned in the Introduction, there is something akin to a magnetic source in Yang-Mills theory.

Another qualitative argument can be offered to make plausible the idea that a magnetic mass should arise. Although I have argued that high-temperature dimensional reduction from four to three dimensions can not be carried out reliably for a gauge theory, one may speculate that there is **some** truth in the idea, when restricted to magnetic (spatial) components of the non-Abelian potential. So one is led, as preliminary to studying the full QCD problem, to an analysis of three-dimensional Euclidean Yang-Mills theory at zero temperature. One quickly discovers that infrared divergences are present in perturbation theory for this model as well, so here again arises the question of a dynamically induced mass. In three dimensions, the coupling constant squared  $g_{(3)}^2$  carries dimensions of mass. (Recall that in a high-temperature reduction  $g_{(3)}$  is related to the four-dimensional coupling  $g$  by  $g_{(3)} = g\sqrt{T}$ .) Therefore it is plausible that three-dimensional Yang-Mills theory generates an  $O(g_{(3)}^2)$  mass, which eliminates its perturbative infrared divergences and suggests the occurrence of an  $O(g^2T)$  magnetic mass in the four-dimensional theory at high  $T$ . Unfortunately thus far no analysis of the three-dimensional Yang-Mills model has led to a proof of such mass generation.

Since the mass is not seen in perturbative expansions, even resummed ones, one attempts a non-perturbative calculation, based on a gap equation. Of course an exact treatment is impossible; one must be satisfied with an approximate gap equation, which effectively sums a large, but still incomplete set of graphs. At the same time, gauge invariance should be maintained; gauge non-invariant approximations are not persuasive.

Deriving an approximate, but gauge invariant gap equation is most efficiently carried out in a functional integral formulation. We begin by reviewing how a one-loop gap equation is gotten from the functional integral, first for a non-gauge theory of a scalar field  $\varphi$ , then we indicate how to extend the procedure when gauge invariance is to be maintained for a gauge field  $A_\mu$ .

Consider a self-interacting scalar field theory (in the Euclidean formulation) whose potential  $V(\varphi)$  has no quadratic term, so in direct perturbation theory one may encounter infrared divergences, and one enquires whether a mass is generated, which would cure them.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + V(\varphi)$$

$$V(\varphi) = \lambda_3 \varphi^3 + \lambda_4 \varphi^4 + \dots$$

The functional integral for the Euclidean theory involves the negative exponential of the action  $I = \int \mathcal{L}$ . Separating the quadratic, kinetic part of  $I$ , and expanding the exponential of the remainder in powers of the field yields the usual loop expansion. As mentioned earlier, the loop expansion may be systematized by introducing a loop-counting parameter  $\ell$  and considering  $e^{-\frac{1}{\ell} I(\sqrt{\ell} \varphi)}$ : the power series in  $\ell$  is the loop expansion. To obtain a gap equation for a possible mass  $\mu$ , we proceed by adding and subtracting  $I_\mu = \frac{\mu^2}{2} \int \varphi^2$ , which of course changes nothing.

$$I = I + I_\mu - I_\mu$$

Next the loop expansion is reorganized by expanding  $I + I_\mu$  in the usual way, but taking  $-I_\mu$  as contributing at one loop higher. This is systematized as in the hard-thermal-loop application with an effective action,  $I_\ell$ , containing the loop counting parameter  $\ell$ , which organizes the loop expansion in the indicated manner:

$$I_\ell = \frac{1}{\ell} \left( I(\sqrt{\ell} \varphi) + I_\mu(\sqrt{\ell} \varphi) \right) - I_\mu(\sqrt{\ell} \varphi)$$

An expansion in powers of  $\ell$  corresponds to a resummed series; keeping all terms and setting  $\ell$  to unity returns us to the original theory (assuming that rearranging the series does no harm); approximations consist of keeping a finite number of terms: the  $O(\ell)$  term involves a single loop.

The gap equation is gotten by considering the self energy  $\Sigma$  of the complete propagator. To one-loop order, the contributing graphs are depicted in the Figure.

$$\Sigma = \text{bubble}(\lambda_3) + \text{loop}(\lambda_4) - \text{triangle}(\mu^2)$$

Self energy resummed to one-loop order.

Regardless of the form of the exact potential, only the three- and four- point vertices are needed at one-loop order; the “bare” propagator is massive thanks to the addition of the mass term  $\frac{1}{\ell} I_\mu(\sqrt{\ell} \varphi) = \frac{\mu^2}{2} \int \varphi^2$ ; the last  $-\mu^2$  in the Figure comes from the subtraction of the same mass term, but at one-loop order:  $-I_\mu(\sqrt{\ell} \varphi) = -\ell \frac{\mu^2}{2} \int \varphi^2$ .

The gap equation emerges when it is demanded that  $\Sigma$  does not shift the mass  $\mu$ . In momentum space, we require

$$\Sigma(p) \Big|_{p^2=-\mu^2} = 0$$

$$\text{Diagram 1} + \text{Diagram 2} \Big|_{p^2=-\mu^2} = \mu^2$$

Graphical depiction of above equation.

While these ideas can be applied to a gauge theory, it is necessary to elaborate them so that gauge invariance is preserved. We shall discuss solely the three-dimensional non-Abelian Yang-Mills model (in Euclidean formulation) as an interesting theory in its own right, and also as a key to the behavior of spatial variables in the physical, four-dimensional model at high temperature.

The starting action  $I$  is the usual one for a gauge field.

$$I = \int d^3x \operatorname{tr} \frac{1}{2} F^i F^i$$

$$F^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$$

While one may still add and subtract a mass-generating term  $I_\mu$ , it is necessary to preserve gauge invariance. Thus we seek a gauge invariant functional of  $A_i$ ,  $I_\mu(A)$ , whose quadratic portion gives rise to a mass. Evidently

$$I_\mu(A) = -\frac{\mu^2}{2} \int d^3x \operatorname{tr} A_i \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) A_j + \dots$$

The transverse structure in the above equation guarantees invariance against Abelian gauge transformations; the question then remains how the quadratic term is to be completed in order that  $I_\mu(A)$  be invariant against non-Abelian gauge transformations. [In fact for the one-loop gap equation only terms through  $O(A^4)$  are needed.]

A very interesting proposal for  $I_\mu(A)$  was given by Nair<sup>15,16</sup> who also put forward the scheme for determining the magnetic mass, which we have been describing. By modifying in various ways the hard thermal loop generating functional (which gives a four-dimensional, gauge invariant but Lorentz non-invariant effective action with a transverse quadratic term), he arrived at a gauge and rotation invariant three-dimensional structure, which can be employed in the derivation of a gap equation.\*

Let me describe Nair's modification. Recall that the hard-thermal-loop generating functional, which I record here again,

$$\Gamma_{\text{HTL}} = \frac{1}{2} \int d\Omega_{\hat{q}} [Q_+^\mu Q_-^\nu A_\mu^a A_\nu^a + S(Q_+^\mu A_\mu) + S(Q_-^\mu A_\mu)]$$

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\*A gap equation for the full gauge-field propagator, rather than just for the mass, has been put forward and analyzed by Cornwall *et al.*; see *Phys. Lett.* **B153**, 173 (1985) and *Phys. Rev. D* **34**, 585 (1986).

is gauge invariant because  $Q_\pm^\mu = (0, \pm \hat{q})$  is light-like and  $S$  is the Chern-Simons eikonal. Before averaging over  $\Omega_{\hat{q}}$  one is dealing with a functional of only  $A_\pm$ ; after averaging all four components of  $A_\mu$  enter. With Nair,<sup>15,16</sup> we observe that another choice for  $Q_\pm^\mu$  can be made, where those vectors remain light-like, but have vanishing time component. This is achieved when the spatial components of  $Q_\pm^\mu$  are complex and of zero length; for example:

$$Q_+^\mu = (0, \mathbf{q}) \quad Q_-^\mu = (0, \mathbf{q}^*)$$

$$\mathbf{q} = (-\cos\theta \cos\varphi - i \sin\varphi, -\cos\theta \sin\varphi + i \cos\varphi, \sin\theta) = \hat{\theta} + i\hat{\varphi}$$

Evidently  $\mathbf{q}^2 = 0$ ,  $Q_+^2 = 0$  and  $Q_- = Q_+^*$ . Using these forms for  $Q_\pm^\mu$  in  $\Gamma_{\text{HTL}}$  still leaves it gauge invariant. Also it is clear that  $\Gamma_{\text{HTL}}$  is real and depends only on the spatial components of the vector potential. Hence this is an excellent candidate for  $I_\mu(A)$ , which therefore, following Nair, we take it to be

$$\left. \frac{\mu^2}{4\pi} \Gamma_{\text{HTL}}(A) \right|_{\substack{\text{evaluated} \\ \text{as above}}}$$

The scheme proceeds as in the scalar theory, except that  $I_\mu(A)$  gives rise not only to a mass term for the free propagator, but also to higher-point interaction vertices. At one loop only the three- and four- point vertices are needed, and to this order the subtracting term uses only the quadratic contribution. Thus the gap equation reads, pictorially

The first three graphs are as in ordinary Yang-Mills theory, with conventional vertices, but massive gauge field propagator (solid line);

$$D_{ij}(p) = \delta_{ij} \frac{1}{p^2 + \mu^2}$$

the first graph depicting the gauge compensating “ghost” contribution, has massless ghost propagators (dotted line) and vertices determined by the quantization gauge, conveniently chosen, consistent with the above form for the propagator, to be

$$\mathcal{L}_{\text{fixing}}^{\text{gauge}} = \frac{1}{2} \nabla \cdot \mathbf{A} (1 - \mu^2 / \nabla^2) \nabla \cdot \mathbf{A}$$

The remaining three graphs arise from Nair’s form for the hard thermal loop-inspired  $I_\mu(A)$ , with solid circles denoting the new non-local vertices. As it happens, the last graph with the four-point vertex vanishes, while the three-point vertex reads

$${}^N V_{ijk}^{abc}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = -\frac{i\mu^2 f^{abc}}{3!(\mathbf{p} \times \mathbf{q})^2} \left\{ \frac{1}{3} \left( \frac{\mathbf{p} \cdot \mathbf{q}}{p^2} + \frac{\mathbf{r} \cdot \mathbf{q}}{r^2} \right) p_i p_j p_k - \frac{\mathbf{r} \cdot \mathbf{p}}{3r^2} (q_i q_j p_k + q_i p_j q_k + p_i q_j q_k) \right\} + 5 \text{ permutations}$$

$$p + q + r = 0$$

The permutations ensure that the vertex is symmetric under the exchange of any pair of



index sets  $(a\ i\ p), (b\ j\ q), (c\ k\ r)$ . Inverse powers of momenta signal the non-locality of the vertex. [We discuss the  $SU(N)$  theory, with structure constants  $f^{abc}$ .]

The result of the computation is

$$\begin{aligned} {}^N\Pi_{ij}^{ab} &= \delta^{ab}\Pi_{ij}^N \\ \Pi_{ij}^N &= \Pi_{ij}^{YM} + \bar{\Pi}_{ij}^N \end{aligned}$$

$\Pi_{ij}^{YM}$  is the contribution from the first three Yang-Mills graphs and  $\bar{\Pi}_{ij}^N$  sums the graphs from  $I_\mu(A)$ . The reported results are<sup>17</sup>

$$\begin{aligned} \Pi_{ij}^{YM}(p) &= N(\delta_{ij} - \hat{p}_i\hat{p}_j) \left[ \left( \frac{-13p^2}{64\pi\mu} + \frac{5\mu}{16\pi} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} - \frac{\mu}{16\pi} - \frac{p}{64} \right] \\ &\quad + N\hat{p}_i\hat{p}_j \left[ \left( \frac{p^2}{32\pi\mu} + \frac{\mu}{8\pi} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} + \frac{\mu}{8\pi} - \frac{p}{32} \right] \\ \bar{\Pi}_{ij}^N(p) &= N(\delta_{ij} - \hat{p}_i\hat{p}_j) \left[ \left( \frac{3p^2}{64\pi\mu} + \frac{3\mu}{16\pi} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} - \frac{p^2}{8\pi\mu} \left( \frac{\mu^2}{p^2} + 1 \right)^2 \frac{\mu}{p} \tan^{-1} \frac{p}{\mu} + \frac{\mu}{16\pi} + \frac{\mu^3}{8\pi p^2} + \frac{p}{64} \right] \\ &\quad - N\hat{p}_i\hat{p}_j \left[ \left( \frac{p^2}{32\pi\mu} + \frac{\mu}{8\pi} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} + \frac{\mu}{8\pi} - \frac{p}{32} \right] \end{aligned}$$

The Yang-Mills contribution  $\Pi_{ij}^{YM}$  is not separately gauge-invariant (transverse) owing to the massive gauge propagators. [At  $\mu = 0$ ,  $\Pi_{ij}^{YM}$  reduces to the standard result<sup>18</sup>:  $N(\delta_{ij} - \hat{p}_i\hat{p}_j) \left( -\frac{7}{32}p \right)$ .] The longitudinal terms in  $\Pi_{ij}^{YM}$  are canceled by those in  $\bar{\Pi}_{ij}^N$ , so that the total is transverse.

$$\Pi_{ij}^N(p) = N(\delta_{ij} - \hat{p}_i\hat{p}_j) \left[ \left( \frac{-5p^2}{32\pi\mu} + \frac{1}{2\pi}\mu \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} - \frac{p^2}{8\pi\mu} \left( \frac{\mu^2}{p^2} + 1 \right)^2 \frac{\mu}{p} \tan^{-1} \frac{p}{\mu} + \frac{\mu^3}{8\pi p^2} \right]$$

[Dimensional regularization is used to avoid divergences.]

Before proceeding, let us note the analytic structures in the above expressions, which are presented for Euclidean momenta, but for the gap equation have to be evaluated at the Minkowski value  $p^2 = -\mu^2 < 0$ . Analytic continuation for the inverse tangent is provided by

$$\frac{1}{x} \tan^{-1} x = \frac{1}{2\sqrt{-x^2}} \ln \frac{1 + \sqrt{-x^2}}{1 - \sqrt{-x^2}}$$

Evidently  $\Pi_{ij}^N(p)$  possesses threshold singularities, at various values of  $-p^2$ .

There is a singularity at  $p^2 = -4\mu^2$  (from  $\tan^{-1} \frac{p}{2\mu}$ ) arising because the graphs in the Figure containing massive propagators describe the exchange of two massive gauge “particles”. Moreover, there is singularity at  $p^2 = -\mu^2$  (from  $\tan^{-1} \frac{p}{\mu}$ ) and also, separately in  $\Pi_{ij}^{YM}$  and  $\bar{\Pi}_{ij}^N$ , at  $p^2 = 0$  (from the  $\pm \frac{p}{64}, \pm \frac{p}{32}$  terms). These are understood in the following way. Even though the propagators are massive, the non-local three-point function contains  $\frac{1}{p^2}$ ,

$\frac{1}{q^2}$ ,  $\frac{1}{r^2}$  contributions, which act like massless propagators. Thus the threshold at  $p^2 = -\mu^2$  arises from the exchange of a massive line (propagator) together with a massless line (from the vertex). Similarly the threshold at  $p^2 = 0$  arises from the massless lines in the vertex (and also from massless ghost exchange). The expressions acquire an imaginary part when the largest threshold,  $p^2 = 0$ , is crossed:  $\Pi_{ij}^{YM}$  and  $\bar{\Pi}_{ij}^N$  are complex for  $p^2 < 0$ .

In the complete answer, the  $p^2 = 0$  thresholds cancel, and the singularity at the  $p^2 = -\mu^2$  threshold is extinguished by the factor  $(\frac{\mu^2}{p^2} + 1)^2$ . Consequently  $\Pi_{ij}^N$  becomes complex only for  $p^2 < -\mu^2$ , and is real, finite at  $p^2 = -\mu^2$ .

$$\Pi_{ij}^N(p) \Big|_{p^2=-\mu^2} = (\delta_{ij} - \hat{p}_i \hat{p}_j) \frac{N\mu}{32\pi} (21 \ln 3 - 4)$$

From the gap equation in the last Figure, the result for the mass is<sup>17</sup>

$$\mu = \frac{N}{32\pi} (21 \ln 3 - 4) \sim 2.384 \frac{N}{4\pi}$$

[in units of the coupling constant  $g_{(3)}^2$  (or  $g^2 T$ ), which has been scaled to unity].

Before accepting this plausible answer for  $\mu$ , it is desirable to assess higher order corrections, for example two-loop contributions. Unfortunately, an estimate<sup>17</sup> indicates that 79 graphs have to be evaluated, and the task is formidable.

An alternative test for the reliability of the above approach and for assessing the stability of the result against corrections has been proposed.<sup>19</sup>

It is suggested that the gap equation be derived with a gauge invariant completion different from Nair's. Rather than taking inspiration from hard thermal loops (which after all have no intrinsic relevance to the three-dimensional gauge theory<sup>†</sup>), the following formula for  $I_\mu$  is taken

$$I_\mu(A) = \mu^2 \int d^3x \operatorname{tr} F^i \frac{1}{D^2} F^i$$

where  $D^2$  is the gauge covariant Laplacian. While ultimately there is no *a priori* way to select one gauge-invariant completion over another, we remark that expressions like the above appear in two-dimensional gauge theories (Polyakov gravity action, Schwinger model) and are responsible for mass generation. If two- and higher- loop effects are indeed ignorable, this alternative gauge invariant completion, which corresponds to an alternative resummation, should produce an answer close to the previously obtained one.

With the alternative  $I_\mu$ , the graphs are as before, where the propagator is still given by the previous expression. However, the three- and four- point vertices in  $I_\mu(A)$  are different. One now finds for the non-local three-point vertex

$$V_{ijk}^{abc}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \frac{-i\mu^2}{3!} f^{abc} (\delta_{ij} \mathbf{q} \cdot \mathbf{r} + q_i p_j) \frac{p_k}{p^2 q^2} + 5 \text{ permutations}$$

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<sup>†</sup>Recall that the hot thermal loop generating functional is related to the Chern-Simons eikonal. Since the Chern-Simons term is a three-dimensional structure, this fact may provide a basis for establishing the relevance of the hard thermal loop generating functional to three-dimensional Yang Mills theory. The point is under investigation by D. Karabali and V. P. Nair.

$$p + q + r = 0$$

and the non-local four-point vertex reads

$$V_{ijkl}^{abcd}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) = \frac{-\mu^2}{4!} f^{abe} f^{cde} \left\{ \frac{1}{2} \delta_{jk} \epsilon_{imn} \epsilon_{lon} \frac{p_m}{p^2} \frac{s_0}{s^2} \right. \\ \left. - \frac{1}{2r^2} \left( \frac{1}{4} \epsilon_{ijm} \epsilon_{klm} - \epsilon_{imn} \epsilon_{kln} \frac{p_m}{p^2} (p - r - s)_j + \epsilon_{imn} \epsilon_{lon} \frac{p_m}{p^2} \frac{s_0}{s^2} (p - r - s)_j (p + q - s)_k \right) \right\} \\ + 23 \text{ permutations}$$

$$p + q + r + s = 0$$

These vertices do not affect the first three graphs in the Figure so that  $\Pi_{ij}^{YM}$  is as before. However, in the last three graphs the alternative non-local vertices produce the following result, with the help of dimensional regularization,

$$\bar{\Pi}_{ij}(p) = N(\delta_{ij} - \hat{p}_i \hat{p}_j) \left( \left( \frac{p^6}{128\pi\mu^5} + \frac{p^4}{32\pi\mu^3} + \frac{7p^2}{64\pi\mu} + \frac{27\mu}{64\pi} - \frac{\mu^3}{16\pi p^2} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} \right. \\ \left. - \left( \frac{p^6}{32\pi\mu^5} + \frac{p^4}{16\pi\mu^3} - \frac{p^2}{16\pi\mu} + \frac{\mu}{32\pi} \right) \left( \frac{\mu^2}{p^2} + 1 \right)^2 \frac{\mu}{p} \tan^{-1} \frac{p}{\mu} \right. \\ \left. - \frac{p^2}{32\pi\mu} - \frac{3\mu}{16\pi} + \frac{49\mu^3}{96\pi p^2} + \frac{\mu^5}{32\pi p^4} + \frac{p^5}{128\mu^4} + \frac{p^3}{32\mu^2} - \frac{p}{16} \right) \\ - N\hat{p}_i \hat{p}_j \left( \left( \frac{p^2}{32\pi\mu} + \frac{\mu}{8\pi} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} + \frac{\mu}{8\pi} - \frac{p}{32} \right)$$

A check on this very lengthy calculation is that summing it with Yang-Mills contributions yields a transverse expression.

$$\Pi_{ij}(p) = N(\delta_{ij} - \hat{p}_i \hat{p}_j) \left( \left( \frac{p^6}{128\pi\mu^5} + \frac{p^4}{32\pi\mu^3} - \frac{3p^2}{32\pi\mu} + \frac{47\mu}{64\pi} - \frac{\mu^3}{16\pi p^2} \right) \frac{2\mu}{p} \tan^{-1} \frac{p}{2\mu} \right. \\ \left. - \left( \frac{p^6}{32\pi\mu^5} + \frac{p^4}{16\pi\mu^3} - \frac{p^2}{16\pi\mu} + \frac{\mu}{32\pi} \right) \left( \frac{\mu^2}{p^2} + 1 \right)^2 \frac{\mu}{p} \tan^{-1} \frac{p}{\mu} \right. \\ \left. - \frac{p^2}{32\pi\mu} - \frac{\mu}{4\pi} + \frac{49\mu^3}{96\pi p^2} + \frac{\mu^5}{32\pi p^4} + \frac{p^5}{128\mu^4} + \frac{p^3}{32\mu^2} - \frac{5p}{64} \right)$$

Another check on the powers of  $\frac{p}{\mu}$  is that the above reduces to the Yang-Mills result at  $\mu = 0$ .<sup>18</sup>

Just as Nair's expression, the present formula exhibits threshold singularities: at  $-p^2 = 4\mu^2$ , which are beyond our desired evaluation point  $-p^2 = \mu^2$ ; there are also threshold singularities at  $-p^2 = \mu^2$ , which are extinguished by the factor  $(\frac{\mu^2}{p^2} + 1)^2$ ; however, those at  $p^2 = 0$  do *not* cancel, in contrast to the previous case — indeed  $\Pi_{ij}(p)$  diverges at  $p^2 = 0$ , and is complex for  $p^2 < 0$ . [It is interesting to remark that the last graph in the last Figure, involving the new four-point vertex, which vanishes in Nair's evaluation, here gives a transverse result with unextinguished threshold singularities at  $-p^2 = \mu^2$  and at  $p^2 = 0$ . The

protective factor of  $(\frac{\mu^2}{p^2} + 1)^2$  arises when the remaining two graphs are added to form  $\bar{\Pi}_{ij}$ , and these also contain non-canceling  $p^2 = 0$  threshold singularities, as does the Yang-Mills contribution.]

Although  $\Pi_{ij}(p)\Big|_{p^2=-\mu^2}$  is finite, it is complex and the gap equation has no solution for real  $\mu^2$ , owing to unprotected threshold singularities at  $p^2 = 0$ , which lead to a complex  $\Pi_{ij}(p)$  for  $p^2 < 0$ .

$$\mu = \frac{N}{32\pi} \left( 29\frac{1}{4} \ln 3 - 22\frac{1}{3} \right) \pm iN \frac{13}{128}$$

$$|\mu| \sim 1.769 \frac{N}{4\pi}$$

It may be that the hot thermal loop-inspired completion for the mass term is uniquely privileged in avoiding complex values for  $-\mu^2 \leq p^2 \leq 0$ , but we see no reason for this.<sup>‡</sup> Absent any argument for the disappearance of the threshold at  $p^2 = 0$ , and reality in the region  $-\mu^2 \leq p^2 < 0$ , we should expect that also the hot thermal loop-inspired calculation will exhibit such behavior beyond the 1-loop order.<sup>§</sup>

Thus until the status of threshold singularities is clarified, the self-consistent gap equation for a magnetic mass provides inconclusive evidence for magnetic mass generation. Moreover, if there exist gauge invariant completions for the mass term, other than the hard thermal loop-inspired one, that lead to real  $\Pi_{ij}$  at  $p^2 = -\mu^2$ , it is unlikely that they all would give the same  $\mu$  at one loop level, which is further reason why higher orders must be assessed.

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<sup>‡</sup>We note that Nair's hot thermal loop-inspired vertex  ${}^N V_{ijk}{}^{abc}$  is less singular than the alternative  $V_{ijk}{}^{abc}$ , when any of the momentum arguments vanish. Correspondingly  $\Pi_{ij}^N(p)$  is finite at  $p^2 = 0$ , in contrast to  $\Pi_{ij}(p)$  which diverges at  $\frac{1}{p^2}$ . However, we do not recognize that this variety of singularities at  $p^2 = 0$  influences reality at  $p^2 = -\mu^2$ ; indeed the individual graphs contributing to  $\Pi_{ij}^N$  are complex at that point, owing to non-divergent threshold singularities at  $p^2 = 0$  that cancel in the sum.

<sup>§</sup>V.P. Nair states that at the two loop level, there is evidence for  $\ln(1 + \frac{p^2}{\mu^2})$  terms, but it is not known whether they acquire a protective factor of  $(\frac{\mu^2}{p^2} + 1)$ .

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